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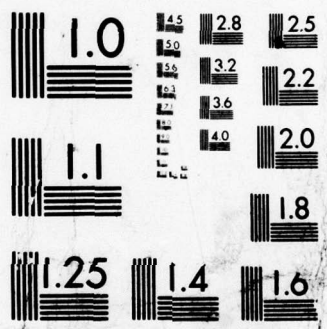
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Contract N004-78-C00636 NR 064-610

Technical Report 2

Report No. GIT-ESM-SNA-14  
(Dec 1978)

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RATE COMPLEMENTARY ENERGY PRINCIPLES; FINITE STRAIN  
PLASTICITY PROBLEMS; AND FINITE ELEMENTS

by

Satya N. Atluri

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RATE COMPLEMENTARY ENERGY PRINCIPLES;  
FINITE STRAIN PLASTICITY PROBLEMS; AND FINITE ELEMENTS

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ABSTRACT:

Complementary energy theorems, for the rate problem of finite strain classical elasto-plasticity, in both Updated and Total Lagrangean rate forms with alternate stress-rates and conjugate measures of strain rate, are studied from the point of view of their application in finite element schemes. Two new complementary theorems, in Updated and Total Lagrangean forms respectively, are proposed. The relative merits of these in application to finite-strain elasto-plastic stress analysis, in the treatment of near-incompressibility at large plastic flow, and in treatment of buckling problems, are briefly discussed.

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1. COMPLEMENTARY PRINCIPLES IN UPDATED LAGRANGEAN (UL) FORM: RATE PROBLEM OF FINITE STRAIN CLASSICAL ELASTO-PLASTICITY:

In this case, the current state  $C_N$ , wherein the cartesian spatial coordinates of a particle are  $y_i^N$  and the true (Cauchy) stress is  $\tau^N$ , is used as a reference state for measuring the rates of variables, in going from  $C_N$  to  $C_{N+1}$ . If  $\dot{\sigma}^*$  is the corotational rate of Kirchhoff stress  $\sigma$  ( $\equiv J\tau$ ;  $J$  is the determinant of the Jacobian);  $\dot{\epsilon}$  is the UL strain rate  $\{\equiv 1/2 [(\nabla^N \dot{u}) + (\nabla^N \dot{u})^T]\}$ ;  $\dot{\omega}$  the UL spin rate  $\{\equiv 1/2 (\nabla^N \dot{u})^T - (\nabla^N \dot{u})\}$ ;  $\nabla^N$  the gradient operator in  $C_N$  ( $\equiv e_i \partial(\ ) / \partial y_i^N$ );  $\dot{u}$  is the rate of displacement;  $\dot{\hat{\sigma}}$  and  $\dot{\hat{\tau}}$ , respectively, are the rates of second and first Piola-Kirchhoff stresses referred to and measured per unit area of  $C_N$ ; it is well-known that:

$$\dot{\hat{\sigma}} = \dot{\sigma}^* - \dot{\epsilon} \cdot \tau^N - \tau^N \cdot \dot{\epsilon}; \quad \dot{\hat{\tau}} = \dot{\tau}^* - \dot{\epsilon} \cdot \tau^N - \tau^N \cdot \dot{\omega} \quad (1.1,2)$$

Further, from the relation for the Jaumann stress tensor  $\tau$  [1], viz.,  $\tau = 1/2 [\dot{\hat{\tau}} \cdot \sigma + \sigma^T \cdot \dot{\hat{\tau}}^T]$ ; we obtain the equation for the UL rate of  $\tau$  as:

$$\dot{\tau} = 1/2 (\dot{\hat{\tau}} + \tau^N \cdot \dot{\omega} + \dot{\omega} \cdot \tau^N + \dot{\hat{\tau}}^T) = \dot{\tau}^* - 1/2 (\dot{\epsilon} \cdot \tau^N + \tau^N \cdot \dot{\epsilon}) \quad (1.3a,b)$$

As noted by Hill [2], the constitutive equation of a bilinear form, for work-hardening elastic-plastic materials, in terms of  $\dot{\sigma}^*$  and  $\dot{\epsilon}$  can be written as:

$$\dot{\sigma}^* = \partial \dot{W} / \partial \dot{\epsilon}; \quad \dot{W} = 1/2 L_{ijkl} \dot{\epsilon}_{ij} \dot{\epsilon}_{kl} - \frac{\alpha}{g} (\lambda_{kl} \dot{\epsilon}_{kl})^2 \quad (1.4a,b)$$

Where  $L_{ijkl}$ ,  $\alpha$ ,  $g$ , and  $\lambda_{kl}$  are as defined in [2]. From Eqs. (1.1-1.4) it is seen that,

$$\dot{\hat{\sigma}} = \partial \dot{W} / \partial \dot{\epsilon}; \quad \dot{W} = \dot{W} - \tau^N: (\dot{\epsilon} \cdot \dot{\epsilon}) \quad (1.5a,b)$$

$$\dot{\hat{\tau}} = \partial \dot{U} / \partial \dot{\epsilon}^T; \quad \dot{U} = \dot{W} - \tau^N: (\dot{\epsilon} \cdot \dot{\epsilon}) + (1/2) \tau^N: (\dot{\epsilon}^T \cdot \dot{\epsilon}) \quad (1.6a,b)$$

$$\dot{\hat{\tau}} = \partial \dot{Q} / \partial \dot{\epsilon}; \quad \dot{Q} = \dot{W} - (1/2) \tau^N: (\dot{\epsilon} \cdot \dot{\epsilon}) \quad (1.7a,b)$$

where  $\dot{\epsilon} = (\nabla^N \dot{u})^T$ . It can then be shown [3,4] that the UL rate form of a complementary energy theorem, with  $\dot{\hat{\sigma}}$  and  $\dot{u}$  as variables, can be stated as the

stationarity condition of the functional,

$$\pi_c^2(\dot{\underline{s}}; \dot{\underline{u}}) = \int_{V_N} \{ -\dot{\underline{s}}^* : \dot{\underline{s}} - (1/2) \underline{\tau}^N : [(\nabla^N \dot{\underline{u}}) \cdot (\nabla^N \dot{\underline{u}})^T] \} dv + \int_{S_{u_n}} \dot{\underline{t}} \cdot \dot{\underline{u}} ds \quad (1.8)$$

with the a priori constraints:

$$\text{Linear Momentum Balance (LMB): } \nabla^N \cdot [\dot{\underline{s}} + \underline{\tau}^N \cdot (\nabla^N \dot{\underline{u}})] + \rho^N \dot{\underline{B}} = 0 \quad (1.9)$$

$$\text{Angular Momentum Balance (AMB): } \dot{\underline{s}} = \dot{\underline{s}}^T \quad (1.10)$$

$$\text{Traction Boundary Condition (TBC): } \underline{n}^* \cdot [\dot{\underline{s}} + \underline{\tau}^N \cdot (\nabla^N \dot{\underline{u}})] \equiv \dot{\underline{t}} = \dot{\underline{t}} \text{ at } S_{\sigma_N} \quad (1.11)$$

and  $\dot{\underline{s}}^*$  is presumed to be obtained through the contact transformation,

$$\dot{\underline{s}}^*(\dot{\underline{s}}) = \dot{\underline{s}} : \dot{\underline{\xi}} - \dot{W}(\dot{\underline{\xi}}) \quad (1.12)$$

and with the Euler-Lagrange Equations (ELE) and Natural Boundary Conditions (NBC):

(i) compatibility,  $\dot{\underline{\xi}} = 1/2 [(\nabla^N \dot{\underline{u}}) + (\nabla^N \dot{\underline{u}})^T]$  and (ii) Displacement B.C (DBC),

$\dot{\underline{u}} = \dot{\underline{u}}$  at  $S_{u_N}$ . The constraint, of TBC is, in general, impossible to satisfy a priori; whereas, it may, in some special circumstances, be possible to satisfy the constraints (1.9 and 1.10). For instance, first noting that  $\dot{\underline{s}} + \underline{\tau}^N \cdot (\nabla^N \dot{\underline{u}}) \equiv \dot{\underline{t}}$ , one may assume a  $\dot{\underline{t}}$  that satisfies LMB  $\nabla^N \cdot \dot{\underline{t}} + \rho^N \dot{\underline{B}} = 0$  (by setting  $\dot{\underline{t}} = \nabla^N \times \underline{\Psi}(y^N) + \dot{\underline{t}}^P$  where  $\dot{\underline{t}}^P$  is any particular solution such that  $\nabla^N \cdot \dot{\underline{t}}^P = -\rho^N \dot{\underline{B}}$ ) and also assume an arbitrary but symmetric  $\dot{\underline{s}}$  field that hence obeys the AMB condition. One can then eliminate displacements from Eq. (1.8) by setting

$$\nabla^N \dot{\underline{u}} = (\underline{\tau}^N)^{-1} \cdot [\nabla^N \times \underline{\Psi} + \dot{\underline{t}}^P - \dot{\underline{s}}] \quad (1.13)$$

Thus, provided that the principal values of  $\underline{\tau}^N$  are not zero (which is a rather special circumstance) one can express Eq. (1.8) in terms of stress-rates only.

On the other hand one can start out assuming  $\dot{\underline{u}}$  and then set,

$$\dot{\underline{s}} = \text{curl curl } \underline{A} + \dot{\underline{s}}^P; \text{ where } \nabla^N \cdot \dot{\underline{s}}^P = -\rho^N \dot{\underline{B}} - \nabla^N \cdot [\underline{\tau}^N \cdot (\nabla^N \dot{\underline{u}})] \quad (1.14)$$

where  $\underline{A}$  are the well-known Maxwell-Morera-Beltrami stress functions and  $\dot{\underline{s}}^P$  is any symmetric particular solution, such as, for instance, given by:

$$\dot{s}_{ij}^P = 0 \ (i \neq j); \ \dot{s}_{ii}^P = \int_{y_i} \nabla^N [-\rho^N \dot{B}_i - (\tau_{kj} \dot{u}_{i;j};_k)] dy_i^N \text{ (no sum on } i) \quad (1.15)$$

However the question of completeness of  $\dot{\underline{s}}^P$  (or the consequences of a priori setting  $\dot{s}_{ij}^P = 0, i \neq j$ ) remains to be answered. Moreover, assuming both  $\dot{\underline{s}}$  and  $\dot{\underline{u}}$  as above has the less desirable consequence that the displacements derived from strain-integrals corresponding to the assumed stresses do not, in general, agree with the assumed displacements. Lastly, one can first assume  $\dot{\underline{u}}$  and then let,

$$\dot{\underline{s}} = \text{curl curl } \underline{A} - \underline{\tau}^N \cdot (\nabla^N \dot{\underline{u}}) + \dot{\underline{s}}^P; \nabla^N \cdot \dot{\underline{s}}^P = -\rho \dot{\underline{N}}_B \quad (1.16)$$

However, the chosen  $\dot{\underline{s}}$  above ceases to be symmetric and the AMB is violated. Eventhough the representations as in Eqs. (1.14,15 & 16) appear to be less than desirable in general, use of such have been made in structural mechanics of curved beams and shells, and their stability, in [5,6].

On the other hand, an alternate general variational theorem, with  $\dot{\underline{t}}, \dot{\underline{e}},$  and  $\dot{\underline{u}}$  as variables, can be stated [3,4] as the condition of stationarity of the functional:

$$\pi_{HW}^{*2}(\dot{\underline{t}}, \dot{\underline{e}}, \dot{\underline{u}}) = \int_{V_N} \{ \dot{\underline{U}}(\dot{\underline{e}}) - \rho \dot{\underline{N}}_B \cdot \dot{\underline{u}} + \dot{\underline{t}}^T : [(\nabla^N \dot{\underline{u}})^T - \dot{\underline{e}}] \} dv - \int_{S_{\sigma N}} \dot{\underline{t}} \cdot \dot{\underline{u}} ds - \int_{S_{u N}} \dot{\underline{t}} \cdot (\dot{\underline{u}} - \dot{\underline{u}}) ds \quad (1.17)$$

which leads to the following ELE and NBC:

$$(LMB): \nabla^N \cdot \dot{\underline{t}} + \rho \dot{\underline{N}}_B = 0; (CC): \dot{\underline{e}} = (\nabla^N \dot{\underline{u}})^T; (DBC): \dot{\underline{u}} = \dot{\underline{u}} \text{ at } S_{u N} \quad ((1.18, 19 \text{ \& } 20))$$

$$(RCL): 2\dot{t}_{ij} = 2\dot{U}/2\dot{e}_{ji} = (L_{ijkl} - \frac{2\gamma}{g} \lambda_{ij} \lambda_{kl}) \frac{(\dot{e}_{kl} + \dot{e}_{lk})}{2} - (\dot{e}_{im} + \dot{e}_{mi}) \tau_{mj}^N - \tau_{il}^N (\dot{e}_{lj} - \dot{e}_{jl}) \quad (1.21)$$

$$(AMB): (\nabla^N \dot{\underline{u}})^T \cdot \underline{\tau}^N + \dot{\underline{t}} = \text{SYMM}; (TBC): \dot{\underline{t}} \equiv (\underline{n}^* \cdot \dot{\underline{t}}) = \dot{\underline{t}} \text{ at } S_{\sigma N} \quad (1.22 \text{ \& } 23)$$

The AMB, Eq. (1.22), is, however, embedded in the structure of  $\dot{\underline{U}}$  when  $\dot{\underline{U}}$  is defined according to Eq. (1.6b). It is also noted that the RCL, Eq. (1.21), cannot, in general, be inverted analytically. Assuming that Eq. (1.24) can be inverted numerically, one can achieve a contact transformation,  $\dot{\underline{\xi}}^T: \dot{\underline{\xi}} - \dot{\underline{U}}$  ( $\dot{\underline{\xi}} = \dot{\underline{T}}^*(\dot{\underline{\xi}})$ ). If, further, one satisfies a priori the Eqs. (1.18 & 23), one obtains a complementary energy principle governed by the stationarity of:

$$\pi_c^{*2}(\dot{\underline{\xi}}) = \int_{V_N} -\dot{\underline{T}}^*(\dot{\underline{\xi}}) dv + \int_{S_{u_N}} \dot{\underline{\xi}} \cdot \dot{\underline{u}} ds \quad (1.24)$$

which has, as its ELE and NBC, Eqs. (1.19 & 20). This principle was first stated by Hill [7]. However, the AMB condition, Eq. (1.22) is not an ELE. Thus, for the principle to be valid, the AMB condition, if at all it is met, must be embedded in the structure of  $\dot{\underline{T}}^*(\dot{\underline{\xi}})$ ; i.e.,  $\dot{\underline{T}}^*(\dot{\underline{\xi}})$  must be such that,  $\dot{\underline{\xi}} + (\partial \dot{\underline{T}}^* / \partial \dot{\underline{\xi}}^T) \cdot \underline{\tau}^N = \dot{\underline{\xi}}^T + \underline{\tau}^N \cdot (\partial \dot{\underline{T}}^* / \partial \dot{\underline{\xi}})$ . It can be shown [4] that this is not true in general; and thus the above is not a valid principle, in general.

To avoid the above difficulties, we transform the functional in Eq. (1.17) to one with  $\dot{\underline{\xi}}; \dot{\underline{u}}; \dot{\underline{\omega}}; \dot{\underline{\xi}}$  as variables. First we note that  $\dot{\underline{\xi}} \equiv \dot{\underline{\xi}} + \dot{\underline{\omega}}$ ; and  $\dot{\underline{U}} = \dot{\underline{Q}} + (1/2)\underline{\tau}^N: (\dot{\underline{\omega}}^T \cdot \dot{\underline{\omega}}) + \underline{\tau}^N: (\dot{\underline{\omega}}^T \cdot \dot{\underline{\xi}})$ . When these substitutions are made, we rewrite the functional in Eq. (1.17) as:

$$\begin{aligned} \pi_{HW}^{*2}(\dot{\underline{\xi}}; \dot{\underline{u}}; \dot{\underline{\xi}}; \dot{\underline{\omega}}) = & \int_{V_N} \{ \dot{\underline{Q}} + (1/2)\underline{\tau}^N: (\dot{\underline{\omega}}^T \cdot \dot{\underline{\omega}}) + \underline{\tau}^N: (\dot{\underline{\omega}}^T \cdot \dot{\underline{\xi}}) - \rho^N \underline{B} \cdot \dot{\underline{u}} + \\ & + \dot{\underline{\xi}}^T: [(\underline{\nabla}^N \dot{\underline{u}})^T - \dot{\underline{\xi}} - \dot{\underline{\omega}}] \} dv - \int_{S_{\sigma_N}} \dot{\underline{\xi}} \cdot \dot{\underline{u}} ds - \int_{S_{u_N}} \dot{\underline{\xi}} \cdot (\dot{\underline{u}} - \dot{\underline{u}}) ds \end{aligned} \quad (1.25)$$

With the constraint that  $\dot{\underline{\omega}}$  must be skew-symmetric. The stationarity of the above functional leads to the ELE and NBC:

$$(RCL) \rightarrow \partial \dot{\underline{Q}} / \partial \dot{\underline{\xi}} = (1/2)(\dot{\underline{\xi}} + \underline{\tau}^N \cdot \dot{\underline{\omega}} + \dot{\underline{\omega}}^T \cdot \underline{\tau}^N + \dot{\underline{\xi}}^T) \equiv \dot{\underline{\xi}} \quad (1.26)$$

$$(LMB) \rightarrow \underline{\nabla}^N \cdot \dot{\underline{\xi}} + \rho^N \underline{B} = 0; (AMB) \rightarrow \underline{\tau}^N \cdot \dot{\underline{\omega}} + \dot{\underline{\xi}} \cdot \underline{\tau}^N + \dot{\underline{\xi}}^T = \text{SYMM.} \quad (1.27 \text{ \& } 28)$$

$$(CC) \rightarrow (\underline{\nabla}^N \dot{\underline{u}})^T = \dot{\underline{\xi}} + \dot{\underline{\omega}}; (TBC) \rightarrow \underline{n}^* \cdot \dot{\underline{\xi}} = \dot{\underline{\xi}} \text{ at } S_{\sigma_N}; \dot{\underline{u}} = \dot{\underline{u}} \text{ at } S_{u_N} \quad (1.29, 30 \text{ \& } 31)$$

It is noted that (AMB) is a clear ELE (corresponding to variation in  $\dot{\underline{\omega}}$ ) and need not be embedded in the structure of  $\dot{\underline{Q}}$ . If we establish the contact transformation, that  $\dot{\underline{\tau}}: \dot{\underline{\epsilon}} - \dot{\underline{Q}} = \dot{\underline{R}}^*(\dot{\underline{\tau}})$  and further if LMB and TBC, Eqs. (1.27 & 30), are satisfied a priori, we can eliminate  $\dot{\underline{u}}$  and  $\dot{\underline{\epsilon}}$  as variables from Eq. (1.25) and obtain the following functional whose stationarity condition leads to a complementary energy theorem:

$$\pi_c^{*2}(\dot{\underline{\tau}}, \dot{\underline{\omega}}) = \int_{V_N} \{ -\dot{\underline{R}}^*(\dot{\underline{\tau}}) + (1/2)\underline{\tau}^N: (\dot{\underline{\omega}}^T \cdot \dot{\underline{\omega}}) - \dot{\underline{\tau}}^T \cdot \dot{\underline{\omega}} \} dv + \int_{S_{u_N}} (\underline{n}^* \cdot \dot{\underline{\tau}}) \cdot \dot{\underline{u}} ds \quad (1.32)$$

Where, by definition,  $\dot{\underline{\tau}}^* = 1/2(\dot{\underline{\tau}} + \dot{\underline{\tau}}^T + \underline{\tau}^N \cdot \dot{\underline{\omega}} + \dot{\underline{\omega}}^T \cdot \underline{\tau}^N)$ . In the above functional the constraints are the LMB, Eq. (1.27) (which is easy to meet by setting  $\dot{\underline{\tau}} = \underline{\nabla}^N \times \underline{\psi} + \dot{\underline{\tau}}^P$ ;  $\dot{\underline{\tau}}^P$  is any particular solution such that  $\underline{\nabla}^N \cdot \dot{\underline{\tau}}^P = -\rho \underline{N}_B$ ), and that  $\dot{\underline{\omega}}$  is a skew-symmetric field [which is easy to meet, by setting  $\omega_{ij} = -e_{ijk}\omega_k$ ]. The ELE and NBC are the AMB, Eq. (1.28); CC, Eq. (1.29); and the DBC, Eq. (1.31). In as much the constraints in the principle are trivial to be met a priori, and the AMB is clearly an ELE, the above complementary principle appears to be the most rational and consistent for practical applications. In applying the above principle in conjunction with a finite element method, the interelement traction reciprocity constraint,  $(\underline{n}^* \cdot \dot{\underline{\tau}})^+ + (\underline{n}^* \cdot \dot{\underline{\tau}})^- = 0$  at  $\rho_{m_N}$ ; can be relaxed a priori, and enforced through Lagrange Multipliers. In doing this, either the well-known hybrid-stress FEM of Pian or the equilibrium model of F. de Veubeke can be used.

\*It is worth noting that the above principle is valid for both isotropic as well as anisotropic materials. In the case of isotropy, it can be shown [4] that the AMB reduces to the constraint:  $\underline{\tau}^N \cdot \dot{\underline{\omega}} + \dot{\underline{\tau}}$  is symmetric. However, even for isotropy, it is more convenient to retain this AMB condition as an ELE corresponding to variations in  $\dot{\underline{\omega}}$ . Thus, when  $\dot{\underline{\tau}}$  (that satisfies LMB), and a skew-symmetric  $\dot{\underline{\omega}}$  are assumed (which, however, need not obey the AMB even for isotropic materials), the above definition for  $\dot{\underline{\tau}}^*$  should be used to recover the AMB as an ELE for either isotropic or anisotropic materials.

## 2. TL RATE COMPLEMENTARY PRINCIPLES: RATE PROBLEM OF FINITE DEFORMATION ELASTO-PLASTICITY:

In this case, the initial (undeformed) state  $C_0$  is used as a reference state in all subsequent considerations. Let  $\underline{\underline{t}}^N$  and  $\underline{\underline{s}}^N$  be the first and second Piola-Kirchhoff stress tensors in  $C_N$  as referred to and measured per unit area in  $C_0$ ; and let  $\underline{\underline{t}}'$  and  $\underline{\underline{s}}'$  be the corresponding TL rates; and let  $\underline{\underline{E}}'$  be the TL rate of strain. It can be shown that,

$$\begin{aligned}\underline{\underline{E}}' &= 1/2(\underline{\underline{e}}' + \underline{\underline{e}}'^T + \underline{\underline{e}}'^T \cdot \underline{\underline{e}}^N + \underline{\underline{e}}^{NT} \cdot \underline{\underline{e}}') ; \text{ where } \underline{\underline{e}}' = (\nabla^0 \underline{\underline{u}})^T ; \underline{\underline{e}}^N = (\nabla^0 \underline{\underline{u}}^N)^T \\ &= (\underline{\underline{F}}^N)^T \cdot \underline{\underline{\xi}} \cdot \underline{\underline{F}}^N \quad [\text{where } \underline{\underline{F}}^N = (\nabla^0 \underline{\underline{Y}}^N)^T] \end{aligned} \quad (2.1a,b)$$

$$\underline{\underline{s}}' = J^N (\underline{\underline{F}}^N)^{-1} \cdot \underline{\underline{s}} \cdot (\underline{\underline{F}}^N)^{-T} ; \underline{\underline{t}}' = J^N (\underline{\underline{F}}^N)^{-1} \dot{\underline{\underline{t}}} = \underline{\underline{s}}' \cdot \underline{\underline{F}}^{NT} + \underline{\underline{s}}^N \cdot \underline{\underline{e}}'^T \quad (2.2 \& 3)$$

$$\underline{\underline{r}}' = 1/2[\underline{\underline{t}}^N \cdot \underline{\underline{\alpha}}' + \underline{\underline{\alpha}}'^T \cdot \underline{\underline{t}}^{NT} + \underline{\underline{t}}' \cdot \underline{\underline{\alpha}}^N + \underline{\underline{\alpha}}^{NT} \cdot \underline{\underline{t}}'^T] \quad (2.4)$$

Further we consider the polar decompositions,  $\underline{\underline{F}}^N = \underline{\underline{\alpha}}^N \cdot (\underline{\underline{I}} + \underline{\underline{h}}^N)$ , where  $\underline{\underline{I}} + \underline{\underline{h}}$  is called the stretch tensor; and  $\underline{\underline{e}}' = \underline{\underline{\alpha}}' \cdot (\underline{\underline{I}} + \underline{\underline{h}}^N) + \underline{\underline{\alpha}}^N \cdot \underline{\underline{h}}'$ . The rigid rotation tensors satisfy the orthogonality conditions,  $\underline{\underline{\alpha}}^N \cdot \underline{\underline{\alpha}}^{NT} = \underline{\underline{I}}$ ; and  $\underline{\underline{\alpha}}'^T + \underline{\underline{\alpha}}' \cdot \underline{\underline{\alpha}}^{NT} = 0$ . Because of Eqs. (2.1b & 2) it is seen that if  $W$  is a potential for  $\underline{\underline{\xi}}$ , a potential  $W'$  can be derived for  $\underline{\underline{s}}$  such that,  $\underline{\underline{s}}' = \partial W / \partial \underline{\underline{E}}'$ . Further, because of Eqs. (2.3 & 4) it is seen that:

$$\underline{\underline{t}}' = \partial U' / \partial \underline{\underline{e}}'^T ; U' = W' + (1/2) \underline{\underline{s}}^N : (\underline{\underline{e}}'^T \cdot \underline{\underline{e}}') \quad (2.5a,b)$$

$$\underline{\underline{r}}' = \partial Q' / \partial \underline{\underline{h}}' ; Q' = W'(\underline{\underline{h}}') + (1/2) \underline{\underline{s}}^N : (\underline{\underline{h}}' \cdot \underline{\underline{h}}') \quad (2.6a,b)$$

From the above, we obtain the contact transformations, that,

$$\underline{\underline{s}}' : \underline{\underline{E}}' - W' = \underline{\underline{S}}'(\underline{\underline{s}}') ; \underline{\underline{t}}'^T : \underline{\underline{e}}' - U' = \underline{\underline{T}}'(\underline{\underline{t}}') ; \underline{\underline{r}}' : \underline{\underline{h}}' - Q' = \underline{\underline{R}}'(\underline{\underline{r}}') \quad (2.7a,b,c)$$

Using procedures analogous to these discussed earlier in connection with the U1 rates case, and as elaborated in [4], the TL rate complementary principle with  $\underline{\underline{s}}'$ , and  $\underline{\underline{u}}$  as variables, can be stated as the stationarity condition of:

$$\pi_c^2(\underline{s}'; \dot{\underline{u}}) = \int_{V_0} \{ \underline{S}'(\underline{s}') + (1/2) \underline{s}'^N : (\underline{e}'^T \cdot \underline{e}') \} dv - \int_{S_{u_0}} \underline{t}' \cdot \dot{\underline{u}} ds \quad (2.8)$$

where,  $\underline{e}'^T = (\nabla^0 \dot{\underline{u}})$ ; and with the constraints: (AMB):  $\underline{s}' = \underline{s}'^T$ ; (LMB):  $\nabla^0 \cdot \underline{s}'^N \cdot \underline{e}'^T + \underline{s}' \cdot \underline{F}^{NT} \} + \rho^0 \underline{B}' = 0$ ; (TBC):  $\underline{n} \cdot \{ \underline{s}'^N \cdot \underline{e}'^T + \underline{s}' \cdot \underline{F}^{NT} \} = \underline{t}' = \bar{\underline{t}}'$  at  $S_{\sigma_0}$ . The ELE and NBC resulting from the stationarity of the above functional are: CC, Eq. (2.1a); and DBC,  $\dot{\underline{u}} = \underline{\dot{u}}$  at  $S_{u_0}$ . The possible ways to satisfy LMB a priori, are: (i) to choose  $\underline{t}'$  that satisfies LMB and in addition to choose a symmetric  $\underline{s}'$ , thereby eliminate  $\underline{e}'$  as a variable, as:  $\underline{s}'^N \cdot \underline{e}'^T + \underline{s}' \cdot \underline{F}^{NT} \equiv \underline{t}' = \nabla^0 \times \underline{\Psi} + \underline{t}'^P$ ; ( $\nabla^0 \cdot \underline{t}'^P = -\rho^0 \underline{B}'$ ) and  $\underline{e}'^T = (\underline{s}'^N)^{-1} \cdot [\nabla^0 \times \underline{\Psi} + \underline{t}'^P - \underline{s}' \cdot \underline{F}^{NT}]$  which requires that the principal values of  $\underline{s}'^N$  be non zero. On the other hand, one can choose  $\dot{\underline{u}}$  in addition to  $\underline{t}'$  that satisfies LMB, and derived  $\underline{s}'$  from them as:  $\underline{s}' = [\nabla^0 \times \underline{\Psi} + \underline{t}'^P - \underline{s}'^N \cdot \underline{e}'^T] \cdot (\underline{F}^{NT})^{-1}$ . However,  $\underline{s}'$  is then unsymmetric in general and hence violates (AMB). In spite of this, there may be situations in structural mechanics where the above representation may be satisfactory; however, with the same drawback, as discussed earlier, in assuming both stresses and displacements.

It can be shown [6], as in the UL rate case, the TL complementary principle, analogous to Eq. (1.24), in terms of  $\underline{t}'$  alone, is invalid in general. Finally, the TL rate complementary principle for finite strain elasto-plasticity, with  $\underline{t}'$  and  $\underline{\alpha}'$  as variables, which is the counterpart of that stated in Eq. (1.32), can be stated as the stationarity of the functional:

$$\pi_c^2(\underline{t}'; \underline{\alpha}') = \int_{V_0} \{ -R^*(\underline{r}') - \underline{t}'^T : [\underline{\alpha}' \cdot (\underline{I} + \underline{h}^N)] - 1/2 \underline{t}'^{NT} : [\underline{\alpha}' \cdot \underline{\alpha}'^{NT} \cdot \underline{\alpha}' \cdot (\underline{I} + \underline{h}^N)] \} dv + \int_{S_{u_0}} \underline{t}' \cdot \dot{\underline{u}} ds \quad (2.9)$$

where  $\underline{r}'$  is defined as in Eq. (2.4), and with the constraints: (i)  $\nabla^0 \cdot \underline{t}' + \rho^0 \underline{B}' = 0$ , which is easy to meet by setting  $\underline{t}' = \nabla^0 \times \underline{\Psi} + \underline{t}'^P$ ; (ii)  $\underline{\alpha}'^N \cdot \underline{\alpha}'^T + \underline{\alpha}' \cdot \underline{\alpha}'^{NT} = 0$  which is also easy to satisfy by writing  $\alpha_{ij} = \alpha_{ij}(\theta_k)$  where  $\theta_k$

are Euler-angles of rigid rotation; and (iii)  $\underline{n} \cdot \underline{\dot{t}}' = \underline{\dot{t}}'$  at  $S_{\sigma_0}$ . The ELE and NBC are (i)  $\text{AMB} \rightarrow \underline{h}' \cdot \underline{\dot{t}}'^N \cdot \underline{\alpha}'^N + (\underline{h}'^N + \underline{I}) \cdot (\underline{\dot{t}}' \cdot \underline{\alpha}'^N + \underline{\dot{t}}'^N \cdot \underline{\alpha}')$  is symmetric; (ii)  $\text{CC} \rightarrow (\underline{\nabla}^0 \underline{\dot{u}})^T = \underline{\alpha}' \cdot (\underline{I} + \underline{h}'^N) + \underline{\alpha}'^N \cdot \underline{h}'$  and the TBC. As before, the above TL rate complementary principle is believed to be the most consistent for practical application.

The consistent complementary principles, as stated in Eqs. (1.32 and 2.9) are useful in (i) obtaining better stress solutions in a numerical analysis of finite strain elasto-plasticity and (ii) in treating situations of near incompressibility as many arise at large plastic flow and (iii) in analysis of stability problems. Numerical evidence for (i) and (ii) appears to be provided in the finite elasticity solutions for compressible as well as incompressible materials reported in [8,9]. In the stability problems, it is seen that the functional,

$$\phi_c^2(\underline{\dot{t}}', \underline{\alpha}') = \int_{V_0} \{ \underline{\dot{R}}'(\underline{\dot{t}}') + \underline{\dot{t}}'^T : [\underline{\alpha}' \cdot (\underline{I} + \underline{h}'^N)] + (1/2) \underline{\dot{t}}'^{NT} : [\underline{\alpha}' \cdot \underline{\alpha}'^{NT} \cdot \underline{\alpha}' \cdot (\underline{I} + \underline{h}'^N)] \} dv \quad (2.10)$$

with the constraints  $\underline{\nabla}^0 \cdot \underline{\dot{t}}' = 0$ ,  $\underline{\alpha}'^N \cdot \underline{\alpha}''^T + \underline{\alpha}' \cdot \underline{\alpha}'^{NT} = 0$ ; and  $\underline{n} \cdot \underline{\dot{t}}' = 0$  at  $S_{\sigma_0}$  attains a zero stationary value at neutral equilibrium. In an FEM procedure this translates into the criterion that the current stiffness matrix becomes singular. If the prebuckling state, say  $C_0$ , is linear, one can make the usual approximations:  $\underline{h} = 0$ ;  $\underline{\alpha} = \underline{I}$ ; and  $\underline{\dot{t}}' = 1/2(\underline{\dot{t}}' + \underline{\dot{t}}'^T + \underline{\alpha}'^T \cdot \underline{\dot{t}}'^0 + \underline{\dot{t}}'^0 \cdot \underline{\alpha}')$ ; and Eq. (2.10) becomes,

$$\phi_c^2(\underline{\dot{t}}'; \underline{\alpha}') = \int_{V_0} \{ \underline{\dot{R}}'(\underline{\dot{t}}') + \underline{\dot{t}}'^T : \underline{\alpha}' + (1/2) \underline{\dot{t}}'^{OT} : (\underline{\alpha}' \cdot \underline{\alpha}') \} dv \quad (2.11)$$

with the constraints  $\underline{\alpha}' = \underline{\alpha}'^T$ ;  $\underline{\nabla}^0 \cdot \underline{\dot{t}}' = 0$ ;  $\underline{n} \cdot \underline{\dot{t}}' = 0$  at  $S_{\sigma_0}$ , and thus leads to an eigen value problem.

#### ACKNOWLEDGEMENTS:

The financial support for this work from the U.S. Office of Naval Research is gratefully appreciated. The author expresses his appreciation to Dr. Nicholas Perrone for his encouragement. Thanks also go to Mrs. T. Rapp for her skillful typing.

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1. REPORT NUMBER <b>14</b> GIT-ESM-SNA-14 <b>TR-2</b>	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) <b>6</b> Rate Complementary Energy Principles; Finite Strain Plasticity Problems; and Finite Elements.		5. TYPE OF REPORT & PERIOD COVERED <b>9</b> Interim Report.
7. AUTHOR(s) <b>10</b> Satya N. Atluri		6. PERFORMING ORG. REPORT NUMBER GIT-ESM-SNA-14
9. PERFORMING ORGANIZATION NAME AND ADDRESS Cooperative for the Advancement of Computational Mechanics - School of Civil Engineering Georgia Institute of Tech., Atlanta, Ga. 30332		8. CONTRACT OR GRANT NUMBER(s) <b>15</b> <del>NO</del> 14-78-C-0636
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, Va. 22217		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 064-610
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) <b>12</b> <b>13p</b>		12. REPORT DATE <b>11</b> June 79
		13. NUMBER OF PAGES 8
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Unlimited		
17. DISTRIBUTION STATEMENT (of this abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES To appear in Proc. of IUTAM Symp. on Variational Methods in Mechanics (S. Nemat-Nasser & K. Washizu, Eds) Pergamon		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Complementary Energy; Finite Elements; Finite Strain Plasticity		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Complementary energy theorems, for the rate problem of finite strain classical elasto-plasticity, in both Updated and Total lagrangean forms with alternate stress-rates and conjugate measures of strain rate, are studied from the point of view of their application in finite elements schemes. Two new complementary theorems, in Updated and Total Lagrangean forms respectively, are proposed. The relative merits of these in applica- tion to finite-strain elasto-plastic stress analysis, in the treatment of <i>→ next page</i>		

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20. Abstract (continued)

near-incompressibility at large plastic flow, and in treatment of buckling problems, are briefly discussed.

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